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A Two-Sex Stable Model Based on Unclassified Birth Data

1. Introduction

IN an earlier paper (Mitra, 1975), it was shown that a population will approach stability with identical intrinsic rate of growth r for the males as well as for the females when the following conditions are met :

(i) $P_M(a)$, the probability of survival from birth to age a for the males and similarly,

$P_F(a')$, the probability of survival from birth to age a' for the females do not change over time, and

(ii) $B_M(a, t) = K_M(a)\, M(a, t)\, u(t)$, where

$B_M(a, t)$ is the number of male births to fathers aged a at time t,

$K_M(a)$ is a function of age a and not of the time t,

$M(a, t)$ is the size of the male population aged a at time t,

$u(t)$ is a measure of the proportion of females to the total reproductive population, and

$B_F(a', t) = K_F(a')F(a', t)v(t)$ is similarly defined for the females.

It was shown that the intrinsic rate of growth r can be obtained as a solution of the integral equation

$$\frac{1}{\int_0^\infty e^{-ra} p_M(a) K_M(a) da} + \frac{1}{\int_0^\infty e^{-ra'} p_F(a') K_F(a') da'} = 1. \tag{1}$$

Interestingly enough, r was found to lie in the interval (r_M, r_F) where r_M and r_F are the intrinsic rates of growth for the males and females, obtained separately from their respective one-sex models, a property that seems to be quite desirable (Coale, 1972) for a stable model based on both sexes. The parameters $u(t)$ and $v(t)$ need not be explicitly specified except to note that by definition, they are positive fractions where the sum

$$u(t) + v(t) = 1 \tag{2}$$

and further, the sex ratio at S, is independent of time. That is to say, the ratio of the total male births

$$B_M(t) = \int_0^\infty B_M(a, t) da \tag{3}$$

to that of the total female births

$$B_F(t) = \int_0^\infty B_F(a', t) da', \tag{4}$$

or

$$\frac{B_M(t)}{B_F(t)} = S, \tag{5}$$

which is independent of time. In terms of the age and sex composition, (5) can be expressed in detail as

$$\frac{u(t)}{v(t)} \frac{\int_0^\infty K_M(a)\, M(a, t) da}{\int_0^\infty K_F(a')\, F(a', t) da'} = S, \tag{6}$$

which shows the variable nature of the ratio $u(t)/v(t)$ with the changes in the age and sex composition over time. It is easily seen from (6), that the range of variation of this ratio is quite narrow, and for all practical purposes therefore,

$u(t)/v(t)$ can virtually be regarded as a constant, though, as will be seen later, such an assumption is not necessary.

The purpose of this paper is to examine a condensed version of this model, that, apart from operational simplicity, can be used on populations that do not suffer from appreciable abnormalities in their sex and age compositions. It is also easy to see that when birth statistics are not decomposed by parents' age, the earlier model cannot be used without necessary adjustments. The formulation of these adjustments and the subsequent modifications of the results are presented in this paper following a theorem which is stated and proved in the next section. For reasons of simplicity this has been done for a one-sex model which has later been extended to take care of both sexes.

2. Sufficient Conditions for Stability

These conditions for a one-sex model have been outlined in the following which are somewhat different from those that are traditional and well known :

(i) First, the probability of survival from birth to age x, namely $p(x)$, is independent of time for all x.

(ii) Second, the ratio of the total number of births at time t to that of the population in any age interval or in any number of age intervals, *connected or disconnected*, *weighted or unweighted*, is also independent of time.

The proof follows in which, for reasons of notational simplicity, only one age interval (α, β) is used with the corresponding population size as

$$\int_{\alpha}^{\beta} P(x, t)\, dx,$$

$P(x, t)$ being the population aged x at time t.

Let the ratio

$$\frac{B(t)}{\int_{\alpha}^{\beta} P(x, t)\, dx} = w \tag{7}$$

be the constant which is independent of time. Noting that

$$P(x, t) = B(t - x)\, p(x), \tag{8}$$

(7) can be rewritten as

$$\frac{B(t)}{\int_{\alpha}^{\beta} B(t-x)\, p(x)\, dx} = w. \tag{9}$$

As is customary, the trial solution

$$B(t-x) = B(t)\, e^{-rx} \tag{10}$$

can be used to simplify (9), producing the integral equation

$$\int_{\alpha}^{\beta} e^{-rx}\, p(x)\, dx = \frac{1}{w}. \tag{11}$$

It can readily be seen that (11) will have only one real root for r that will dominate the population trajectory as $t \to \infty$. Of special interest is the case when (α, β) encompasses the entire age span for which w will be equal to the intrinsic birth rate or the birth rate of the stable population.

3. Condensed Model

For developing a condensed two-sex model let the total male births $B_M(t)$ be assumed as proportional to the size of the male population $M(t)$ and also to a proportion of female to total population $u(t)$ of reproductive ages. $M(t)$ thus represents the size of the male population within some conveniently defined interval and $u(t)$, as before, need not be explicitly specified except to note that conditions (2) and (5) are met. Thus,

$$B_M(t) = K_M\, M(t)\, u(t), \tag{12}$$

and

$$B_F(t) = K_F\, F(t)\, v(t), \tag{13}$$

where K_M and K_F are the appropriate constants of proportionality. It follows that

$$\frac{B_M(t)}{K_M\, M(t)} + \frac{B_F(t)}{K_F\, F(t)} = u(t) + v(t) = 1, \tag{14}$$

which is true for all t. Noting that

$$M(t) = \int_{\alpha_M}^{\beta_M} B_M(t-a)\, p_M(a)\, da, \tag{15}$$

and

$$F(t) = \int_{\alpha_F}^{\beta_F} B_F(t-a')\, p_F(a')\, da', \tag{16}$$

(14) can be expressed as

$$\frac{B_M(t)}{K_M \int_{\alpha_M}^{\beta_M} B_M(t-a)\, p_M(a)\, da} + \frac{B_F(t)}{K_F \int_{\alpha_F}^{\beta_F} B_F(t-a')\, p_F(a')\, da'} = 1 \tag{17}$$

and then, because of (5), as

$$B_F(t)\left[\frac{1}{K_M \int_{\alpha_M}^{\beta_M} B_F(t-a)\, p_M(a)\, da} + \frac{1}{K_F \int_{\alpha_F}^{\beta_F} B_F(t-a')\, p_F(a')\, da'}\right] = 1. \tag{18}$$

Using a trial solution

$$B_F(t-a) = B_F(t)\, e^{-ra}, \tag{19}$$

(18) is reduced to

$$\frac{1}{K_M \int_{\alpha_M}^{\beta_M} e^{-ra}\, p_M(a)\, da} + \frac{1}{K_F \int_{\alpha_F}^{\beta_F} e^{-ra'}\, p_F(a')\, da'} = 1. \tag{20}$$

Defining $w_M(0)$ and $w_F(0)$ for the two sexes in the same manner as w in (7) for $t = 0$, i.e.,

$$K_M u(0) = w_M(0), \tag{21}$$

and

$$K_F v(0) = w_F(0), \tag{22}$$

(20) can also be expressed as

$$\frac{u(0)}{w_M(0) \int_{\alpha_M}^{\beta_M} e^{-ra}\, p_M(a)\, da} + \frac{v(0)}{w_F(0) \int_{\alpha_F}^{\beta_F} e^{-ra'}\, p_F(a')\, da'} = 1. \qquad (23)$$

As before, equations (20) or (23) will have only one real root of r, that will provide a measure of the intrinsic rate of growth for the two-sex model. The necessary conditions for the two-sex stable model can thus be specified for this condensed version as :

(i) unchanging values of $p_M(a)$ and $p_F(a')$; and

(ii) $B_M(t) = K_M\, M(t)\, u(t)$ and $B_F(t) = K_F\, F(t)\, v(t)$;

where the parameters have been defined in the foregoing.

In order to ascertain the boundary of r, it may be noted that for $r = r_M$, the denominator of $u(0)$ in (23) becomes equal to one, while that of $v(0)$ is less than one for $r_M > r_F$ and greater than one for $r_M < r_F$. Accordingly, for $r = r_M$, the l.h.s. of (23) is greater than one for $r_M > r_F$ and less than one for $r_M < r_F$ and conversely for $r = r_F$. Thus, the interval (r_M, r_F) includes r.

4. The Solution of r

For the solution of r, the denominator of $u(0)$ in (23) is first written as

$$e^{-(r-r_M)\bar{a}}\, w_M(0) \int_{\alpha_M}^{\beta_M} e^{-(r-r_M)(a-\bar{a})}\, e^{-r_M a}\, p_M(a)\, da, \qquad (24)$$

where $\bar{a}$ is the average age of the frequency function

$$w_M(0)\, e^{-r_M a}\, p_M(a)\, da \qquad (25)$$

defined over the interval (α_M, β_M). The factor $e^{-(r-r_M)\bar{a}}$ in (24) is the moment generating function (m.g.f.) of the same frequency function with argument $r - r_M$, moments being calculated around the mean. The m.g.f. can be written as

$$1 + \frac{(r - r_M)^2}{2!}\, V_M(a), \qquad (26)$$

in which $V_M(a)$, the variance of the function is much smaller compared to its factor $(r - r_M)^2/2!$ so that terms of $O(r^2)$ can be ignored in (26). A close approximation of (24) is thus provided by

$$e^{-(r-r_M)\bar{a}}. \tag{27}$$

Using similar simplification for the denominator of $v(0)$, (23) can be approximated by

$$u(0)\ e^{(r-r_M)\bar{a}} + v(0)\ e^{(r-r_F)\bar{a}'} = 1. \tag{28}$$

Expansion of the exponentials in (28) yields

$$u(0)(r - r_M)\bar{a} + v(0)(r - r_F)\bar{a}' = 0 \tag{29}$$

after terms of the $O(r^2)$ are neglected as before. It may be pointed out that an equation similar to (29) can also be obtained for classified birth data and that, when compared with the expression obtained earlier (Mitra, 1975), seems to provide a better approximation of r. In its present form,

$$r = r_M + \frac{r_F - r_M}{1 + \dfrac{u(0)}{v(0)}\ \dfrac{\bar{a}}{\bar{a}'}}. \tag{30}$$

5. The Choice of $(u)0$ and $(v)0$

It may first be noted that as the population approaches stability, the parameters $u(t)$ and $v(t)$ also approach their respective limiting values u and v. The relationship between u and $u(0)$ and similarly that between v and $v(0)$ will be investigated next prior to the selection of a specific set of values. Denoting the limiting values of the w functions by w_M and w_F, the following equalities

$$\frac{u(0)}{w_M(0)} = \frac{u}{w_M}, \tag{31}$$

and

$$\frac{v(0)}{w_F(0)} = \frac{v}{w_F} \tag{32}$$

can be generated by rearranging terms in (21) and (22) and noting the independence of K_M and K_F over time. In that case, (23) can be rewritten as

$$\frac{u}{w_M \int_{\alpha_M}^{\beta_M} e^{-ra} p_M(a)\, da} + \frac{v}{w_F \int_{\alpha_F}^{\beta_F} e^{r-a'} p_F(a')\, da'} = 1, \tag{33}$$

in which the expressions are all written in terms of the intrinsic or the limiting values of the parameters. It is easy to see that the denominators of both u and v are equal to unity because of (11), and also because r is the common intrinsic rate of growth for the two sexes. Thus,

$$w_M \int_{\alpha_M}^{\beta_M} e^{-ra} p_M(a)\, da = 1, \tag{34}$$

which because of (31) can also be expressed as

$$\frac{u}{u(0)} w_M(0) \int_{\alpha}^{\beta} e^{-ra} p_M(a)\, da = 1. \tag{35}$$

The factor $u/u(0)$ in (35) is the denominator of $u(0)$ in (23) which was approximated by (27). Accordingly, (35) simplifies into

$$\frac{u}{u(0)} e^{-(r-r_M)\bar{a}} = 1, \tag{36}$$

so that

$$u(0) = u[1 - (r - r_M)\bar{a}], \tag{37}$$

and similarly,

$$v(0) = v[1 - (r - r_F)\overline{a'}] \tag{38}$$

can be obtained by neglecting terms of $O(r^2)$. Adding (37) with (38) and simplifying,

$$r = r_M + \frac{r_F - r_M}{1 + \frac{u}{v} \frac{\bar{a}}{\overline{a'}}} \tag{39}$$

an expression similar to (30) is obtained. Both expressions will, of course, produce identical value of r when $u/v = u(0)/v(0)$, and the apparent equality is due to the fact that terms of $O(r^2)$ are ignored, which also confirms the inference drawn earlier about the limited variation of $u(t)/v(t)$. This distinction, though subtle and relatively unimportant when put into practice, is made from mathematical points of view.

Returning to practical usage, the notion of stability in a population may be regarded as the manifestation of a balance in the age and sex composition, according to which a symmetrical relationship may be postulated between the two sexes. This sounds logical, especially in monogamous societies, in which the inequality between u and v cannot be justified without strong supporting evidence. In any event, information about the u function is all that is needed to estimate the relevant parameters and unless otherwise indicated, the assumption that $u = v$ seems quite appropriate.

6. Concluding Observations

It is interesting to note additional simplification in the algebraic solution of r, when $u = v$. Substituting $u = v$ in (39)

$$r = r_M + \frac{r_F - r_M}{1 + \bar{a}/\bar{a}'}, \tag{40}$$

the value of which is then substituted in (37) and (38) in order to estimate $u(0)$ and $v(0)$. These in turn produce estimates of K_M and K_F from (12) and (13). The intrinsic birth rates $b_M(TS)$ and $b_F(TS)$ where TS stands for the two-sex model, may be obtained from the standard equations, namely,

$$b_M(TS) = \frac{1}{\int_0^\infty e^{-ra}\, p_M(a)\, da}, \tag{41}$$

and

$$b_F(TS) = \frac{1}{\int_0^\infty e^{-ra'}\, p_F(a')\, da'}, \tag{42}$$

where the limits of the integrals are changed from the respective reproductive intervals to cover the entire age range. An interesting observation may be made in this context, that this model, condensed as it is, can generate most but not all of the characteristics of the stable population. There are, however, a few exceptions, examples of which are net reproduction rates, lengths of generations etc. These cannot be estimated in the absence of specific rates, even though the intrinsic rate of growth, related as it is to those functions can be obtained.

A few words about the estimation procedure of r seem to be in order. The solution (40) apparently depends on the knowledge of r_M, r_F, $\bar{a}$ and $\bar{a'}$ where the latter four parameters are to be obtained from somewhat unconventional expressions. In fact, r_M is to be solved from an integral equation like (11), which is

$$\int_{\alpha_M}^{\beta_M} e^{-r_M a}\, p_M(a)\, da = \frac{M(0)}{B_M(0)} = \frac{1}{w_M(0)}. \tag{43}$$

Note the similarity between (43) and the conventional integral equation for the standard one-sex stable population model, namely,

$$\int_0^\infty e^{-rx}\, \phi(x)\, dx = 1, \tag{44}$$

where $\phi(x)$ is the net maternity function, and the solution of r is provided by

$$r = \frac{\log_e R_0}{T}, \tag{45}$$

in which R_0 is the net reproduction rate and T is the length of a generation. Similarly, r_M can be obtained when in (44) $\phi(x)$ is substituted by $w_M(0)\, p_M(a)$ and x by a. The average age $\bar{a}$ defined by

$$\bar{a} = \int_{\alpha_M}^{\beta_M} a e^{-r_M a}\, p_M(a)\, da \Bigg/ \int_{\alpha_M}^{\beta_M} e^{-r_M a}\, p_M(a)\, da \tag{46}$$

may next be obtained. The values of r_F and $\bar{a'}$ may similarly be derived from appropriate equations, and these together with r_M and $\bar{a}$, will determine r from (40).

Before closing, it may be mentioned that the value of r obtained from this model depends on the choice of $M(t)$ and $F(t)$ in (12) and (13). One may use the total number of males in the age interval, say, 15-54 as a measure of $M(t)$ and similarly, the females in the age interval 15-49 may be accepted for $F(t)$.

The intervals may be varied, or designated multipliers may be used to adjust these numbers, or some form of weighted sum of the populations by age-groups like the denominator of the sex-age adjusted birth rate (UN, 1956), may also be considered. The computational formulas will have to be adjusted accordingly, although the basic nature of the algebraic relationships among the parameters will not be affected by any of such procedures.

References

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